Any answer that involves a design for a Finite Automaton should contain information about the following five components of the FA (corresponding to the 5-tuple description):

i) The set of states $Q$; ii) the alphabet $\Sigma$; iii) the start state; iv) the set of final states $F$; v) the set of transitions $\delta$, which can be either shown in the form of a state diagram (preferred) or a transition table.

There will be five problems, each of which is equally weighted.

1. Write formal descriptions of the following sets.

   a) The set containing the numbers 1, 10, and 100

   b) The set containing all integers that are greater than 5

   c) The set containing all natural numbers that are less than 5

   d) The set containing the string aba

   e) The set containing the empty string

   f) The set containing nothing at all

2. If $C$ is a set with $c$ elements, how many elements are in the power set of $C$? Explain your answer.
3. Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first $n$ natural numbers, and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first $n$ cubes. Prove the following equalities by induction on $n$, to arrive at the curious conclusion that $C(n) = S^2(n)$ (that is $S(n)$, squared) for every $n$.

(a) $S(n) = \frac{1}{2}n(n + 1)$.

(b) $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n + 1)^2$.

4. Give a Finite Automaton for each of the following languages defined over the alphabet $\Sigma = \{0, 1\}$:

a) $L = \{ w \mid w$ contains the substring 101 $\}$

b) $L = \{ w \mid w$ ends in 001 $\}$

c) $L = \{ w \mid w$ has a 1 in its 2nd last position, if such a position exists $\}$

5. Give a Finite Automaton for the following language over the alphabet $\Sigma = \{0, 1, 2\}$:

$L = \{ w \mid$ the sum of the symbols in $w$ is a multiple of 3 $\}$

For example, 021201 is part of the language because the sum of all its symbols equals 6 ($6 \mod 3 = 0$); whereas, 010012 is not in the language because it sums up to 4 ($4 \mod 3 = 1$).